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Efficient Evaluation of Multidimensional Time-Varying Density Forecasts with an Application to Risk Management

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Abstract

We propose two simple evaluation methods for time varying density forecasts of continuous higher dimensional random variables. Both methods are based on the probability integral transformation for unidimensional forecasts. The first method tests multinormal densities and relies on the rotation of the coordinate system. The advantage of the second method is not only its applicability to any continuous distribution but also the evaluation of the forecast accuracy in specific regions of its domain as defined by the user's interest. We show that the latter property is particularly useful for evaluating a multidimensional generalization of the Value at Risk. In simulations and in an empirical study, we examine the performance of both tests.

Keywords: Multivariate Density Forecast Evaluation; Probability Integral Transformation; Multidimensional Value at Risk; Monte Carlo Simulations.

JEL classification: C52; C53.

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1. Introduction

Evaluation of the accuracy of forecasts occupies a prominent place in the finance and economics literature. However, most of this literature (e.g., Diebold and Lopez, 1996) focuses on the evaluation of point forecasts as opposed to interval or density forecasts. The driving force for this over-focus is that, until recently, point forecasts appeared to serve well the requirements of the forecast users. However, there is increasing evidence that a more comprehensive approach is needed. One example is Value at Risk (VaR) which is defined as the maximum loss on a portfolio over a certain period of time that can be expected with a certain probability. When returns are normally distributed, the VaR of a portfolio is a simple function of the variance of the portfolio.[1] In this case, normality justifies the use of point forecasts for the variance. However, when the return distribution is non-normal, as is now the general consensus, the VaR of a portfolio is determined not just by the portfolio variance but by the entire conditional distribution of returns. More generally, decision making under uncertainty with asymmetric loss function and non-Gaussian variables involves density forecasts (see Tay and Wallis, 2000; and Guidolin and Timmermann, 2005, for a survey and discussion of density forecasting applications in finance and economics).

The increasing importance of forecasts of the entire (conditional) density naturally raises the issue of forecast evaluation. The relevant literature, although developing at a fast

[1] When the mean return on an asset is assumed to be zero, as is commonly the case in practice when dealing with short-horizon returns, the VaR of a portfolio is simply a constant multiple of the square root of variance of the portfolio.

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pace, is still in its infancy. This is somewhat surprising considering that the crucial tools employed date back a few decades. Indeed, a key contribution by Diebold et al. (1998) relies on the probability integral transformation (PIT) result in Rosenblatt (1952). Diebold et al. point out that the correct density is weakly superior to all forecasts. This suggests that forecasts should be evaluated in terms of their correctness as this is independent of the loss function. To this end, Diebold et al. (1998) employ the PIT of the univariate density forecasts which, if accurate, are $i.i.d.$ standard uniform. They measure the forecast accuracy by the distance between the empirical distribution of the PITs and the 45° line and argue that the visual inspection of this distance may provide valuable insights into the deficiencies of the model and ways of improving it. Obviously, standard goodness-of-fit tests (see Noceti et al., 2003 for a comparison of the existing goodness-of-fit tests) can be directly applied to the PITs and additional tests have been proposed by Anderson et al. (1994), Li (1996), Granger and Pesaran (1999), Berkowitz (2001), Li and Tkacz (2001), Hong (2001), Hong and Li (2003), Bai (2003), Corradi and Swanson (2006) and Hong et al. (2007).

The existing evaluation methods of the multidimensional density forecasts (MDF) rely on the advances made in the univariate case. Diebold et al. (1999) extend the PIT idea to the multivariate forecasts by factoring the multivariate probability density function (PDF) into its conditionals and computing the PIT for each conditional. As in the univariate case, the PIT of these forecasts is $i.i.d.$ uniform if the sequence of forecasts is correct. Clements and Smith (2000, 2002) extend Diebold et al.'s idea and propose two tests based on the product and ratio of the conditionals and marginals. While the latter tests

perform well when there is correlation misspecification, they underperform the original test by Diebold et al. (1999) when such misspecification is absent. However, both approaches rely on the decomposition of each period forecasts into their conditionals which may be impractical for some applications (e.g., for numerical approximations of density forecasts).

Other approaches concerning the evaluation of multivariate density forecasts have been proposed by Sarno and Valente (2004) and Chen and Fan (2006). They, however, are concerned with superior predictive ability of two competing forecast models. The test proposed by Sarno and Valente, which is the equivalent of the test of Diebold and Mariano (1995) in the context of density forecasting, relies on the integrated square difference. Chen and Fan on the other hand, forecast the joint densities via semi-parametric copula models and employ the Kullback-Leibler Information Criterion (KLIC) to discriminate between them. Dick et al. (2008) and Li and Xu (2009) employ the KLIC framework to evaluate the forecasts of the joint density of exchange rates.

Similar to Diebold et al. (1998, 1999) and Clements and Smith (2000, 2002), this paper assumes that the forecasting model is correct under the null hypothesis. This assumption has important implications which impact upon the evaluation tools employed (see Corradi and Swanson, 2006). However, as the focus of this paper is to relate our test to similar tests, we ignore parameter estimation error and potential dynamic misspecification but acknowledge that these could be important. Finally, we stress that

forecasts may vary over time making a forecast evaluation based on the laws of large numbers unfeasible.

The outline of the remainder of this paper is as follows. In Section 2, we discuss an evaluation procedure for multinormal density forecasts. Section 3 presents a test for arbitrary continuous densities while Section 4 discusses the results of Monte Carlo simulations and an empirical application for the newly proposed tests. Finally, Section 5 concludes.

2. Evaluation Procedure for Multinormal Density Forecasts

Rosenblatt (1952) showed that for the cumulative distribution function (CDF) $\widehat{F}_t$ (PDF $\widehat{f}_t$), which correctly forecasts the true data generating process (DGP) F_t of the observation x_t, i.e., for which $\widehat{F}_t(x_t) = F_t(x_t)$, the PIT

$$z_t = \int_{-\infty}^{x_t} \widehat{f}_t(u)du = \widehat{F}_t(x_t)$$

is *i.i.d.* according to $U[0,1]$. Therefore, the adequacy of forecasts can be easily evaluated by examining the z_t series for violations of independence and uniformity.

The PIT idea is extended to the multivariate case by Diebold et al. (1999). Their test procedure (D-test hereafter) factors each period MDF into the product of the conditionals

$$\widehat{f}_{t-1}(x_{1t}, x_{2t}, ..., x_{Nt}) = \widehat{f}_{t-1}(x_{Nt} \mid x_{1t}, x_{2t}, ..., x_{N-1,t}) \cdots \widehat{f}_{t-1}(x_{2t} \mid x_{1t}) \cdot \widehat{f}_{t-1}(x_{1t})$$

and obtain the PIT for each conditional distribution, producing a set of N z series, which are *i.i.d*. $U[0,1]$ individually and as a whole whenever the MDF is correct. [2] Rejecting the null of *i.i.d*. $U[0,1]$ for any, as well as the combined z_t series implies that the MDF is misspecified. Clements and Smith (2000, 2002) propose two tests (CS-tests hereafter) based on the product (CS1) and the ratio (CS2) of PITs for the conditionals and marginals, where the N dimensional vector of scores has typical elements $z_t^j = z_{2|1,t}^c \cdot z_{1,t}^m$ and $z_t^j = z_{2|1,t}^c / z_{1,t}^m$ respectively.

For a multinormal density forecast, we describe below a test (MN-test hereafter) that avoids the possibly cumbersome factorization of the MDF. Instead, we transform the coordinate system according to a linear transformation composed of a translation and a rotation and compute the PITs for each marginal distribution. Note that the standard multinormality tests (e.g., Cox and Small, 1978; Smith and Jain, 1988) do not apply for time varying distributions.

Specifically, let $X_t = (X_{1,t},...,X_{N,t})$ represent an N-dimensional multinormal random variable with mean μ_t and the variance-covariance matrix Σ_t. The null hypothesis assumes that the MDF $\widehat{F}_{t-1}$ is the same as the true distribution F_t of X_t and we do not distinguish between these functions in what follows. It is well known that the random variable $\tilde{\tilde{X}}_t = R_t(X_t - \mu_t)$, where R_t is the matrix of eigenvectors of Σ_t, is multinormal

[2] There are $N!$ different ways to factor the joint density forecast $\widehat{f}_t(x_{1,t-1},...,x_{N,t-1})$, giving us a wealth of z series with which to evaluate the forecast.

with mean zero and a diagonal variance-covariance matrix $\tilde{\Sigma}_t = R_t \Sigma_t R_t^T$. Since X_t is multinormal, $\tilde{X}_t$ is a collection of independent univariate variables with marginal distributions $\tilde{F}_{1,t},...,\tilde{F}_{N,t}$, $\tilde{F}_{i,t} \sim \mathrm{N}(0, \tilde{\Sigma}_t(i,i))$. Moreover, the null hypothesis that the observations x_t are drawn from $\hat{F}_{t-1}$ is equivalent to the hypothesis that the transformed observations $\tilde{x}_t = R_t(x_t - \mu_t)$ are drawn from $\tilde{X}_t$. From the results in Rosenblatt (1952) and by the independence of the components of $\tilde{X}_t$ follows then, that the scores $\tilde{z}_{i,t} = \tilde{F}_{i,t}(\tilde{x}_{i,t})$, $i = 1,...,N$, are independently and uniformly[3] distributed on $[0,1]^N$ individually and as a whole. As the scores are computed from N independent marginals, their computation simplifies to the multiplication of *N* unidimensional PITs, with the important implication that the computation time increases only linearly in *N*. In the next section, we show that linear transformations re-emerge as a useful tool in a test that does not rely on the normality of the forecasts.

3. Evaluation Procedure for Arbitrary Continuous MDFs

The test introduced in this section (Q-test hereafter) fulfils two purposes. On the one hand, it is a simple, ready-to-use procedure to evaluate an arbitrary (continuous) MDF. On the other hand, it allows for focusing on a specific region of the MDF instead of examining it over its entire domain. As we shall explain later in this section, existing tests can then be used to verify the region-specific accuracy of the forecasts. The latter application is

[3] This will be the case when all variables in $\tilde{X}_t$ are not degenerated. Otherwise, we use only variables with positive variance to compute the scores.

particularly interesting from a risk-management perspective. Risk managers and regulators are interested, generally, in the likelihood of large losses, i.e. in a specific tail of the distribution. If this is the case, then, a model superior in forecasting the central part of the distribution will be eschewed in favor of another model which accurately forecasts the tails. This objective motivates the censored likelihood test of Berkowitz (2001), in which the observations not falling into the negative tail of the distribution (with cut-off point being decided by the user's requirements) are truncated.

As the Q-test is based on the PIT computation, we show first in a simple example that for a correct MDF $\widehat{F}_{t-1}$, the PITs $\widehat{F}_{t-1}(x_t)$ are not necessarily uniformly distributed. For the standard binormal $\widehat{F}_{t-1}(x_t)$, it is straightforward to compute that the probability mass of the contour area $\{y \in R^2 : \widehat{F}_{t-1}(y) < 0.025\}$ is 0.117. Thus, under this distribution, the probability of obtaining a score $z_t = \widehat{F}_{t-1}(x_t) < 0.025$ is 0.117 rather than 0.025 as would be the case if z_t were uniformly distributed. It follows that, generally, the multidimensional extension of the PIT does not produce uniformly distributes scores. However, a simple modification in the PIT computation restores the uniformity. First, we transform the series $x = \{x_t\}_{t=1}^{T}$ into $x_t^M = Max\{x_{1,t},....,x_{N,t}\} \cdot (1,....,1)$ and then compute the scores $z_t^M = \widehat{F}_{t-1}(x_t^M)$. Instead of the original observation x_t, we use for the computation of the PIT the projection of the largest coordinate of x_t on the main diagonal along the vector perpendicular to the corresponding axis (see Figure 1). Note that for unidimensional forecasts, our procedure reduces to the traditional PIT. In the appendix, we prove the following result.

Proposition 1: If $\{F_t\}_{t=1}^T$ is the DGP for the sequence $\{x_t\}_{t=1}^T$, then $\{z_t^M = F_t(x_t^M)\}_{t=1}^T$, $x_t^M = Max\{x_{1,t},....,x_{N,t}\}\cdot(1,....,1)$, is *i.i.d.* according to the uniform distribution $U[0,1]$.

The proposition leads to a simple test for the MDF accuracy that verifies the uniformity of the z_t^M-scores (see Noceti et al., 2003). For an intuition of the proof, we focus on two-dimensional orthants (quadrants) $Q((v,v)) = \{y \in R^2 : y \le (v,v)\}$, $v \in R$, as illustrated by the dark gray rectangle in Figure 1.[4] The crucial observation is that for any point x_t inside (outside) of the quadrant $Q((v,v))$, x_t^M also lies also inside (outside) of $Q((v,v))$. In other words, $x_t \le (v,v)$ implies $x_t^M \le (v,v)$ and $x_{i,t} > v$ for at least one i implies $x_t^M > (v,v)$. As a consequence, the probability of obtaining a score $z_t^M = \widehat{F}_{t-1}(x_t^M)$ below $\widehat{F}_{t-1}((v,v))$ is equivalent to the probability of x_t lying in $Q((v,v))$, i.e., it is equal to $\widehat{F}_{t-1}((v,v))$.

[Figure 1]

The proposed procedure effectively transforms a multidimensional MDF $\widehat{F}_{t-1}$ into a unidimensional random variable $Z_t^M = \widehat{F}_{t-1}(X_t^M)$, $X_t^M = Max\{X_{1,t},...., X_{N,t}\}\cdot(1,....,1)$. Due to the Max{.} operator, each realization z_t^M of Z_t^M exploits the information in the entire multidimensional observation x_t. Forecast $\widehat{F}_{t-1}$ is deemed correct whenever the proportion

[4] Strictly speaking, the set $Q((v,...,v)) = \{y \in R^N : y \le (v,...,v)\}$ is an orthant in the coordinate system centred at $(v,...,v)$. Due to the importance of orthants (quadrants), we call our procedure the Q-test**.**

of observations that fall into each orthant $Q((v,...,v))$ approximates the probability of this orthant under $\widehat{F}_{t-1}$. In particular, the Q-test allows for assessing the accuracy of the forecasts in the "negative tail" of the distribution, as illustrated in the following application to risk management.

Multidimensional Value at Risk

In a market with *N* assets, an investor is interested in the event *E* that the random return of each asset falls below a certain value *v*. Equipped with the forecast $\widehat{F}_{t-1}$, the investor can compute v_t such that $\widehat{F}_{t-1}((v_t,...,v_t)) = \alpha$, i.e., such that the event *E* is expected to occur with probability α. If the value of v_t is negative, the investor can compute the loss due to the event *E* for any portfolio of long positions.

The rationale in this example lies at the heart of the concept of Value at Risk (VaR) which is now one of the most widely used risk measure among practitioners, largely due to its adoption by the Basel Committee on Banking Regulation (1996) for the assessment of the risk of the proprietary trading books of banks and its use in setting risk capital requirements (see Jorion, 2000). For the unidimensional CDF $\widehat{F}_{t-1}$, the VaR at the coverage level 1-α is the quantile v_t for which $\widehat{F}_{t-1}(v_t) = \alpha$. Generalizing this definition to the MDF $\widehat{F}_{t-1}$, we require that the multidimensional VaR (MVaR) $(v_t,...,v_t)$ satisfies the condition

$\widehat{F}_{t-1}((v_t,...,v_t)) = \alpha$.[5] From the definition $z_t^M = \widehat{F}_{t-1}(x_t^M)$ follows immediately that z_t^M is less than α whenever all components of the observation $x_t = (x_{1,t},...,x_{N,t})$ fall below (exceed) the critical value v_t,

$$z_t^M < \alpha \Leftrightarrow x_{i,t} < v_t \text{ for all } i = 1,\ldots,N$$

The latter property has important consequences when assessing the MVaR forecasts (the density forecasts for an orthant $Q((v_t,...,v_t))$). For a sufficiently large number of observations, we can compute the proportion of scores that exceed the MVaR (the proportion of observations that fall into $Q((v_t,...,v_t))$), and compare this number to the nominal significance level α. We refer to this procedure as unconditional accuracy. On the other hand, the conditional accuracy requires that the number of scores that exceed the MVaR forecast should be unpredictable when conditioned on the available information (i.e., the MVaR violations should be serially uncorrelated). To assess both types of accuracy, we can resort to the unconditional accuracy test of Kupiec (1995) and the conditional accuracy test of Christoffersen (1998), which have been developed for testing the VaR accuracy. Although both tests are designed for univariate densities, they still apply for our purposes because the Q-test effectively converts a MDF into a univariate score variable.

[5] Asymmetric specifications of MVaR, $\widehat{F}_{t-1}((v_{1,t},...,v_{N,t})) = \alpha$, where $v_{1,t} \neq ... \neq v_{N,t}$, are also possible and can be evaluated with the Q-test in a suitably transformed coordinate system.

In the context of the last example, the MVaR is a suitable instrument of risk measurement for situations of joint losses incurred by long positions in N assets. If, however, the investor contemplates also (some) short positions, she will be interested in the joint risk of negative and positive returns. In other words, the investor will be interested in the appropriate orthant which combines negative returns for the long positions and positive returns for the short positions. The accuracy of the density forecasts for areas other than the "negative orthant" can be assessed by transforming the canonical coordinate system. In order to compute the z_t^M -scores in the transformed system, we have to express the observations x_t and the arguments in the MDF $\widehat{F}_{t-1}$ in the new coordinates. Specifically, for a translation vector μ_t and a rotation matrix R_t, we compute $\tilde{x}_t = R_t(x_t - \mu_t)$, $\tilde{x}_t^M = Max(\tilde{x}_{1,t},...,\tilde{x}_{N,t}) \cdot (1,...,1)$ and $\tilde{z}_t^M = \tilde{F}_t(\tilde{x}_t^M) = \widehat{F}_{t-1}(R_t^{-1}\tilde{x}_t^M + \mu_t)$. Note that under this transformation, $\tilde{F}_t$ is a CDF and the $\tilde{z}_t^M$ -scores are *i.i.d.* $U[0,1]$ when $\widehat{F}_{t-1}$ is the true DGP. The orthant $Q((v_t,...,v_t))$ in the transformed system corresponds then to a different area of the original $\widehat{F}_{t-1}$ domain and the accuracy of the $\widehat{F}_{t-1}$ in this area can be tested by the same means as in the canonical system. Figure 2 shows the example of $N = 2$ assets with means zero and the MDF $\widehat{F}_{t-1}$. The rotation of the coordinates clockwise by 90° relocates the south-east orthant (a positive and a negative return) in the canonical coordinates to the south-west orthant (two negative returns). The investor can, consequently, assess the MVaR under $\widehat{F}_{t-1}$ for a portfolio composed of a short position in the first asset and a long position in the second asset.

[Figure 2]

The possibility of generating scores in different coordinate systems allows, potentially, for gathering abundant information on the tested MDF. Unlike the D-test and CS-tests, where various independent score series can be generated, the scores in the Q-test are not independent across transformations. Figure 3 shows the scatter plot of the scores computed under the standard binormal in the canonical (x-axis) and in the 90°-rotated system (y-axis) are highly dependent. For example, both scores are not less than 0.2 simultaneously.

[Figure 3]

On the other hand, the use of only one score series raises the question of the transformation that maximizes the power of the test. A simple transformation that, arguably, comes closest to this goal, projects the largest component from the principal component analysis of the covariance matrix Σ_t of $\widehat{F}_{t-1}$ on the main diagonal. This transformation can be constructed by rotating the demeaned $\widehat{F}_{t-1}$ firstly by the matrix of eigenvectors of Σ_t, and then by the matrix that rotates the axis with the largest variance to the main diagonal.

4. Monte Carlo Simulations and Empirical Results

Although a comprehensive study of the statistical properties of the proposed tests is beyond the scope of this work, we performed Monte Carlo simulations, in which we compared the performance of four test procedures (D-test, CS-tests, Q-test and MN-test).

In the first experiment, we generated observations according to a mixture of two binormal distributions, i.e., at each time *t*, an observation was drawn from one of the distributions according to the probability weights in the mixture. Note that this experiment can be interpreted as emulating a time-varying DGP that is forecasted correctly by time-varying densities. Specifically, we used two mixtures, $\frac{1}{2}N((-\delta,-\delta), I) + \frac{1}{2}N((\delta,\delta), I)$ and $\frac{1}{2}N((0,0),((1,-\delta/2),(-\delta/2,1))) + \frac{1}{2}N((0,0),((1,\delta/2),(\delta/2,1))$, where δ is interpreted as the deviation from the null hypothesis. The scatter plots of the representative data are reproduced in Figure 4 and Figure 5, respectively.

[Figure 4 and 5]

For both mixtures, we tested the null hypothesis that the observations came from a binormal with mean μ and variance Σ, both estimated from the relevant sample. In order to compute the test statistic in the D-test and the CS-tests, we factor the multinormal pdf $f(x;\mu,\Sigma)$ into a product of two multinormal pdfs (a conditional and a marginal),

$$f(x;\mu,\Sigma) = f(x_1;\mu_{x_1|x_2},\Sigma_{x_1|x_2})\, f(x_2;\mu_2,\Sigma_{22}), \qquad (1)$$

where,

$$x=(x_1,x_2), \quad \mu=(\mu_1,\mu_2), \quad \Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix},$$

$$\mu_{x_1|x_2} = \mu_{x_2} + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - x_1), \quad \Sigma_{x_1|x_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

In our bivariate case, we computed one score for the marginal $f(x_2;\mu_2,\Sigma_{22})$ and another for the conditional $f(x_1;\mu_{x_1|x_2},\Sigma_{x_1|x_2})$ pdf for each observation $(x_{1,t},x_{2,t})$. When the null is true, these scores are *i.i.d.* $U[0,1]$ (Diebold et al., 1999). Two mutually independent scores can be also obtained from another factorization, in which x_1 and x_2 are swapped but they are not independent from the scores obtained in the first factorization. Therefore, we use one pair of independent scores per observation in the evaluation of the D-test and the CS-tests. For the Q-test, only one independent score series can be generated. For the reasons discussed at the end of Section 3, we compute the scores under the transformation that projects the largest component from the principal component analysis of the covariance matrix Σ on the main diagonal. Finally, the MN-test produces, by construction, two independent score series.

Table 1 reports the results of the experiment for two data generating processes (mixture 1 and 2) and different values of the parameter δ. The table contains the p-values of the Pearson's goodness-of-fit χ^2-statistic for all tests that is computed from 2500 data points under the null of binormality with the parameters estimated from the sample.

[Table 1]

The performance of all tests, with the exception of the CS2-test and – to a lesser extent – the D-test, is comparable for the first mixture despite the fact that the Q-test uses only half of the scores relative to the other tests. For the second mixture, however, the Q-test

and CS-tests clearly outperform their competitors.[6] The comparative disadvantage of the latter is due to the fact that the covariance matrices, estimated from the samples, are close to the identity matrix. In this case, the null hypothesis takes the form of the standard binormal. The D-test and the MN-test verify then, whether the marginal distributions follow the univariate standard normal and ignore the correlation between the variables. The Q-test and the CS-tests, on the contrary, combine the information from both variables, which allows for a sharper detection of a deviation from binormality. Furthermore, we found in this experiment that the performance of the Q-test does not deteriorate essentially in the canonical coordinate system.

Regarding the effect of the dimension N on the power of the tests, we investigated in another simulation the extent to which the tests suffer from the curse of dimensionality. For this purpose, we generalized the mixture 1 from the previous example to $\frac{1}{2}N((-\delta/\sqrt{N},...,-\delta/\sqrt{N}),I) + \frac{1}{2}N((\delta/\sqrt{N},...,\delta/\sqrt{N}),I)$. In this mixture, δ is the Euclidean distance between the origin of the coordinates and the means $(\pm\delta/\sqrt{N},...,\pm\delta/\sqrt{N})$ of the DGP. This distance remains constant for all dimensions N which makes the test results comparable across dimensions,

$$d((\delta/\sqrt{N},\ldots,\delta/\sqrt{N}),(0,..,0)) = d((-\delta/\sqrt{N},\ldots,-\delta/\sqrt{N}),(0,..,0)) = \sqrt{(\delta/\sqrt{N})^2 N} = \delta$$

[6] These results confirm the findings in Clements and Smith (2002) for the CS-tests and the D-test.

As in the previous experiment, the scores were computed under the null of multinormality with parameters estimated from the relevant sample. For reasons of computational efficiency, the scores in the Q-test were obtained in the coordinate system rotated by the matrix of the eigenvectors of the estimated covariance matrix. As the hypothesised function becomes then a product of *N* marginal PDFs, the computation simplifies to the multiplication of *N* PITs of these marginals. This operation can be performed efficiently in higher dimensions. For the evaluation of the MN-test, we stacked the *N*-dimensional scores into a single vector. Additionally, in an unidimensional version of the MN-test (MN1-test hereafter), we examined the vector of MN scores that corresponded to the rotated variable with the largest variance (the first principal component). The N vectors of scores in the D-test were obtained from the repeated application of the factorization (1) to the N-dimensional forecast. One score per observation $(x_{1,t},...,x_{N,t})$ was then computed for each of the independent factors. Table 2 reports the p-values of the Pearson's χ^2 -statistic for the tests Q/MN1/MN/D as computed from a sample of 2500 observations drawn from the above mixture for each value of δ and *N*.

[Table 2]

The MN1-test is by far the most powerful among the three contenders and seems to retain power in higher dimensions, at least for the parameter space under study. Interestingly, the tests MN and D are the worst performing ones in spite of exploiting *N*-1 additional independent score series relative to the tests MN1 and Q. Further analysis of the MN scores showed that the information on the true DGP is concentrated in the scores

corresponding to the first principal component. The inclusion of other scores dilutes this information and leads to the loss of power. For the D-test, none of the N individual score vectors is consistently superior to any other or to the stacked vectors. Finally, the Q-test performs worse than MN1-test but is clearly more powerful than the tests MN and D, although its power appears to decrease somehow with higher *N*.

Finally, in an empirical study, we tested the hypothesis of multinormal distribution for the daily returns of S&P500, Dow Jones and Nasdaq equity indices. Table 3 presents summary statistics for the continuously compounded daily return series of equity indices computed from the raw prices. The mean returns are almost identical for all series and close to zero. In line with previous evidence, the distribution of daily returns is heavily leptokurtic and the hypothesis of univariate normality is strongly rejected for each equity index.

[Table 3]

In light of the individual results for the three indices, it comes as no surprise that the null of multinormality, where the parameters are estimated from the sample, is strongly rejected by all three tests with the p-values of the Pearson's χ^2-test virtually equal to zero.[7] More interesting are the insights offered by the scores computed by the Q-test. As explained in Section 3, these scores allow for verifying the accuracy of the forecasted density in specific areas. For the Q-test in the canonical system, the scores contain

[7] For brevity, the detailed results are not presented. They are available from the authors upon request.

information on the forecast accuracy in the "negative orthants" of the distribution. Table 4 contains the proportion of scores that fell into the orthant $Q((v_t,...,v_t))$, where v_t is defined by $\widehat{F}_{t-1}((v_t,...,v_t)) = \alpha$ for the nominal significance levels α= 0.005, 0.01, 0.015, 0.02 and 0.025. By the results presented in Section 3, this proportion is equal to the exceedence rate of the MVaR at the corresponding coverage level 1-α. These proportions (exceedence rates) are consistently higher than the nominal levels α which means that the number of observations far in the negative tails is higher than that implied by a multinormal distribution. The stylized fact of fat tails in financial time series seems to be valid also in the multidimensional context.

[Table 4]

5. Summary and Conclusion

The focus of the forecasting literature has recently shifted to interval and density forecasts. This shift has been motivated by applications in finance and economics as well as the realization that density and interval forecasts convey more information that point estimates. Density forecasts naturally raise the question of evaluation. While efficient evaluation techniques for the univariate case have developed rapidly, the literature on multivariate density forecast evaluation remains limited. Indeed, the Diebold et al. (1999) PIT test remains the main reference with extensions proposed by Clements and Smith (2000, 2002). A drawback of these approaches is that they rely on the PDF factorization into conditionals and marginals which may prove challenging even for simple functions.

In this paper, we provide flexible and intuitive alternative tests of multivariate forecast accuracy that rely on the univariate PIT idea and avoid the cumbersome decomposition into conditionals and marginals. We performed Monte Carlo simulations and an empirical case study that exemplified the applications of both procedures. Finally, regarding the sources of forecast errors, we expect the parameter estimation uncertainty to be of second-order importance when compared to dynamic misspecification (Chatfield, 1993). However, shedding light on the power of the proposed test in the presence forecast inaccuracy requires formal investigation which may suggest a possible avenue for future research.

6. Appendix

Proof of Proposition 1:

For a series of T observations $x=\{x_t\}_{t=1}^{T}$, $x_t=(x_{1,t},...,x_{N,t})$ of random variables $\{X_t\}_{t=1}^{T}$ with continuous distributions $\{F_t\}_{t=1}^{T}$, we define the series of T transformed values $\{z_t^M=F_t(x_t^M)\}_{t=1}^{T}$, where $x_t^M = Max\{x_{1,t},....,x_{N,t}\}\cdot(1,...,1)$, and the corresponding random variables $Z_t^M = F_t(X_t^M\} = F_t(Max\{X_{1,t},....,X_{N,t}\}\cdot(1,...,1))$.

We observe that if x_t belongs to the orthant $Q((v,...,v)) =\{y\in R^N : y\le(v,...,v)\}$, $v\in R$, then x_t^M also belongs to $Q((v,...,v))$. This follows from the fact that $x_{i,t}\le v$ for i=1,…,N implies $Max\{x_{1,t},....,x_{N,t}\}\le v$. On the other hand, if x_t does not belong to $Q((v,...,v))$ then there must exist $x_{i,t}>v$ and, hence, $x_t^M\notin Q((v,...,v))$. Therefore,

$$\forall x_t \in Q((v,...v)),\ F_t(x_t) \le F_t(x_t^M) \le F_t((v,...v)), \qquad \text{(A1)}$$

$$\forall x_t \notin Q((v,...v)), F_t(x_t^M) > F_t((v,...v)).$$

In order to prove that Z_t^M is uniformly distributed over $U[0,1]$, we have to show that $\Pr(Z_t^M < \alpha) = \alpha$. From (A1) follows that $z_t^M = F_t(x_t^M) \le \alpha =: F_t((v,...,v))$ whenever $x_t \in Q((v,...,v))$. The probability of the latter event is equal to the density mass over $Q((v,...,v))$, i.e., equal to $F_t((v,...,v)) = \alpha$.

Finally, since $Z_t^M \sim U[0,1]$ for any CDF $\widehat{F}_t$, the distribution of Z_t^M is independent of the distribution of Z_s^M for any s≠t.

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Table 1 The performance of Q/MN/D/CS1/CS2 in a Monte Carlo Simulation

δ	**Mixture 1** ½N((-δ,-δ),I) + ½N((δ, δ),I)	**Mixture 2** ½N((0,0),((1, δ/2),(δ/2,1))) +½ N((0,0),((1,-δ/2),(-δ/2,1)))
0.60	.430/.253/.308/.321/.961	.834/.242/.356/.341/.749
0.80	.072/.006/.251/.092/.545	.632/.702/.481/.546/.723
1.00	.003/.002/.197/.000/.728	.181/.093/.199/.132/.943
1.20	.000/.000/.128/.000/.535	.017/.349/.284/.004/.204
1.40	.000/.000/.000/.000/.130	.000/.432/.391/.000/.009
1.60	.000/.000/.000/.000/.094	.000/.000/.432/.000/.000

Notes: The table reports the p-values of the Pearson's χ^2-test for the tests Q/MN/D/CS1/CS2, respectively, under the null N(μ, Σ) with parameters μ and Σ estimated from 2500 realizations of the corresponding mixture. The test statistic was computed from 5000 stacked scores (499 degrees of freedom) for MN/D/C1/C2 and from 2500 scores (249 degrees of freedom) for the Q-test.

Table 2 The performance of Q/MN1/MN/D in a Monte Carlo Simulation

δ	N								
	2	3	4	5	6	7	8	9	10
1.0	.025	.122	.071	.277	.423	.501	.514	.697	.329
	.004	.007	.233	.020	.197	.007	.021	.200	.010
	.073	.245	.779	.092	.774	.931	.707	.435	.217
	.604	.473	.329	.749	.231	.793	.893	.583	.438
1.2	.008	.015	.065	.245	.558	.321	.195	.078	.291
	.000	.000	.000	.002	.010	.000	.005	.000	.007
	.036	.065	.269	.129	.671	.727	.342	.812	.775
	.543	.139	.891	.393	.551	.173	.515	.741	.116
1.4	.000	.015	.295	.074	.039	.347	.412	.358	.060
	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.000	.001	.413	.708	.299	.387	.047	.551	.214
	.373	.569	.298	.905	.542	.259	.233	.972	.491
1.6	.000	.000	.000	.000	.002	.020	.139	.002	.098
	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.000	.000	.002	.312	.249	.551	.003	.191	.606
	.148	.631	.721	.337	.612	.638	.914	.285	.733
1.8	.000	.000	.000	.000	.000	.000	.000	.091	.037
	.000	.000	.000	.000	.000	.000	.000	.000	.000
	.000	.000	.017	.064	.143	.194	.248	.124	.322
	.004	.120	.348	.573	.940	.341	.089	.777	.483

Notes: The p-values of the Pearson's χ^2-statistic for the tests Q/MN1/MN/D, respectively, under the null of multinormality with the parameters estimated in the sample of 2500 N-dimensional observations, drawn from the mixture ½N($(-\delta/\sqrt{N},...,-\delta/\sqrt{N}), I$) + ½N($(\delta/\sqrt{N},...,\delta/\sqrt{N}), I$) . The χ^2-statistics were computed from 2500 scores (249 degrees of freedom) for the tests Q and MN1 and from 2500*N scores (250*N-1 degrees of freedom) for the tests MN and D.

Table 3 Summary Statistics

Statistics	S&P500	Dow Jones	Nasdaq
Mean (%)	0.0083	0.0147	0.0128
Stand Dev (%)	1.1389	1.0919	1.8163
Skewness	0.051	-0.064	0.116
Kurtosis	4.984	6.004	6.614
χ^2 **-stat (df=249)**	433.5(0)	378.1(0)	514.8(0)

Notes: The table reports the mean, standard deviation, skewness, kurtosis and the Pearson's χ^2 statistic (p-values in parenthesis) under the null of normality for the log returns for S&P500, Dow Jones and Nasdaq for the sample period 25/09/1998 to 29/08/08 (2498 daily observations).

Table 4 MVaR Unconditional Forecast Accuracy for the Multinormal Density

Nominal Significance	*%x*	t_u
$\alpha = 0.5\%$	0.881	2.037
$\alpha = 1\%$	1.361	1.558
$\alpha = 1.5\%$	1.962	1.664
$\alpha = 2\%$	2.562	1.778
$\alpha = 2.5\%$	3.163	1.892

Notes: The table reports the percentage of exceptions out of 2498 daily observations (i.e., the proportion of times the forecasted MVaR is exceeded) and the Kupiec's *t*-statistic to test the null hypothesis of unconditional accuracy for different nominal significance levels.

Figure 1: The contour area $\{y \in R^2 : \widehat{F}_{t-1}(y) < 0.025\}$ (gray) and the quadrant $Q((-1,-1)) = \{y \in R^2 : y \leq (-1,-1)\}$ (dark gray) for the standard binormal $\widehat{F}_{t-1}$. For observations (black dots) lying inside (outside) of the quadrant $Q((-1,-1))$, the "highest" of the projections on the main diagonal along the axes lies also inside (outside) of $Q((-1,-1))$.

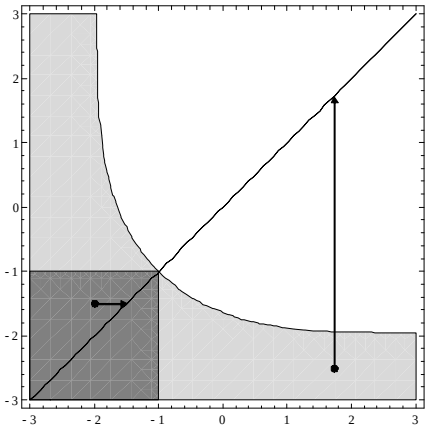

Figure 2: After the rotation of the canonical system clockwise by 90°, the south-east orthant Qse moves to the south-west position Qsw. The dashed lines are the main diagonals in the original and the rotated system while the shaded ellipse is the contour area of $\widehat{F}_{t-1}$.

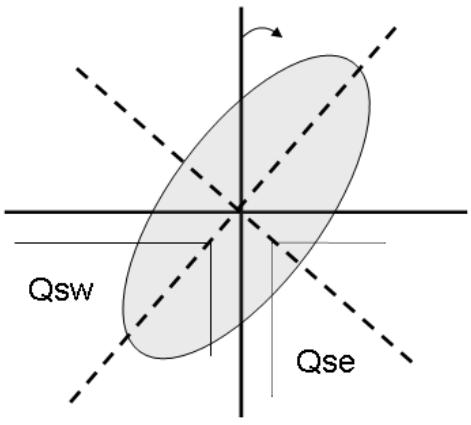

Figure 3: A scatter plot of scores generated from 1000 standard binormal observations under the null N((0,0),I). The x-axis (y-axis) corresponds to the scores computed in the canonical (90°-rotated) system.

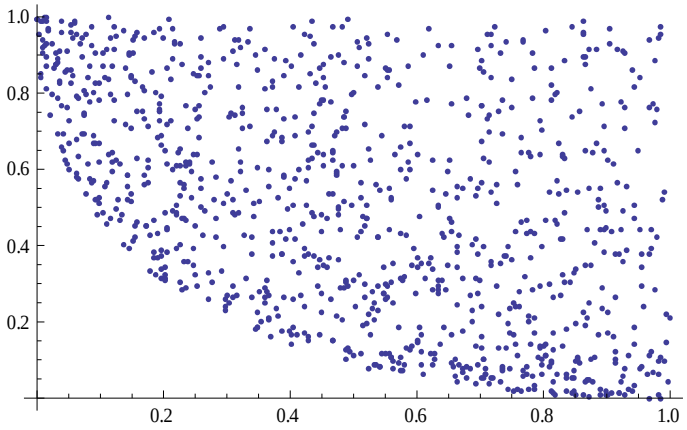

Figure 4: A sample of 1000 observations from the mixture 1: ½ N((-δ,-δ),I) + ½N((δ,δ),I)) for δ=1.4.

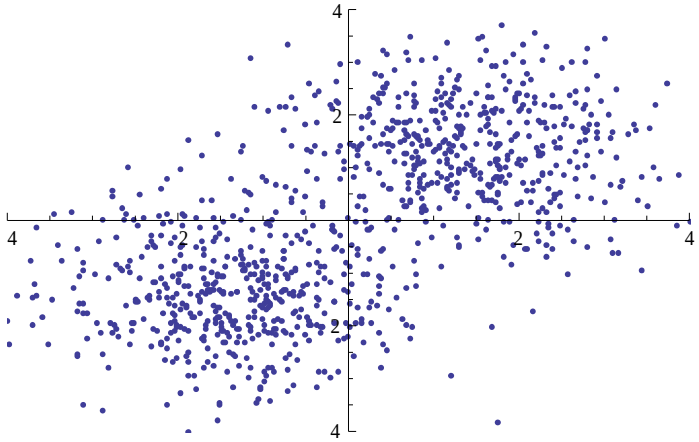

Figure 5: A sample of 1000 observations from the mixture 2: ½N((0,0),((1,-δ/2), (-δ/2,1))) + ½N((0,0), ((1, δ/2),(δ/2,1))) for δ=1.